SCHRIFTELIJK TENTAMEN
COSMOLOGY
1^{st} term 2007/2008

NOTE: THIS EXAM CONTAINS 5 QUESTIONS, ONE QUESTION PER PAGE.

YOU ONLY HAVE TO COMPLETE 4 QUESTIONS OF CHOICE !!!!

Question 1.

The cosmological principle says that the Universe is isotropic and homogeneous. At first this sounds far from convincing, the world around us is far from isotropic and homogeneous.

a) What are the four fundamental forces of nature ? Provide information on their relative strength. Which force dominates the evolution and fate of the Universe ? Why would the weakest force be able to play the main violin in the Universe ?

b) Write down the Einstein field equation, and explain the significance of each part and constituent of the equation. Why is the GR description such a fundamental and revolutionary step wrt. the old Newtonian theory ?

c) What is the importance of the cosmological principle within the context of the General Theory of Relativity ?

a) Name the 3 possible geometries of a uniform space and provide a comparative listing of some of their basic properties (parallel lines, sum of angles triangle, circumference circle, and a few more).

d) Give an inventory of around five pieces of evidence for the isotropy of the Universe on scales of Gigaparsec. Give a short explanation of each of them and indicate in a sketch about which objects' spatial distribution in the Universe they inform us.

e) Give an inventory of four pieces of evidence for the homogeneity of the Universe on scales of Gigaparsecs. Give some short explanations, possibly with sketches (and even equations).

Question 2.

We are going to investigate the Robertson-Walker metric.

a) The Robertson-Walker metric is the general expression for the geometry of a uniform space-time. Write down the expression for a space with coordinate distances r and time t, and explain the significance of the various terms. Also, give the expressions for the curvature terms $S_k(r)$ for the various values of k.

b) What is cosmological redshift ? Derive the relation between redshift z and expansion factor $a(t)$ of the Universe.

c) While coordinate distance r is mainly a theoretical concept, redshift z is what is measured by a telescope. Show that one can translate a redshift z to the coordinate distance of an object by

$$r(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{(H(z')/H_0)}$$

Given the expression for $r(z)$ one also can derive the expressions for $S_k(r)$. How are these essential relations for observational cosmology called ? (after the person who solved the integrals analytically).

d) Give the proper definitions for the two cosmological quantities that describe the first order derivative of the expansion, $\dot{a}$, and the second order derivative $\ddot{a}$. That is, give the definitions for the *HubbleparameterH*(t) and the *accelerationparameterq*.

e) While the correct and full expression for the relation between r and redshift z can only be found once one has a model for the dynamics of the Universe (ie. the Friedman models), it is possible to find a relation between them up to a few orders in redshift. Show that

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2$$

with the index "0" indicating the current cosmological epoch.

f) The proper distance between us and an object that emitted at time t_e the radiation that we observe now at time t_0, is given by

$$d_p(t_0) \equiv c \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

Show that up to second order

$$d_p(t_0) = c(t_0 - t_e) + \frac{cH_0}{2}(t_0 - t_e)^2$$

g) Show that the redshift z of the same object is given by

$$z \approx H_0(t_0 - t_e) + \frac{1 + q_0}{2} H_0^2 (t_0 - t_e)^2$$

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h) Combining g) and h), derive the final result, that between coordinate distance $d_p(t_0)$ and redshift z,

$$d_p(t_0) = \frac{c}{H_0} z \left\{ 1 - \frac{1+q_0}{2} z \right\}$$

This is the second order version of the Hubble diagram, with the linear part representing the linear Hubble law.

Question 3.

a) Write down the full Friedman equations, including cosmological constant and including pressure. Describe and explain the various quantities in the equations.

b) One may derive a Newtonian equivalent of these equations. Describe how. Identify the three *fundamental* factors in the FRW equations that you will not find in the Newtonian equivalent.

c) What is the definition of the critical density ρ_{crit} ? Derive the expression for ρ_{crit},

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

Given the best current value for $H_0 = 71$km/s/Mpc, what is the present-day value of the critical density. Both in terms of g/cm^3 and in $M_\odot$/Mpc^3.

d) What is the definition of Ω ? Derive the relation between curvature kc^2/R_0^2 and Ω.

e) The 2 Friedmann equations include a third equation (which is also one of the 10 Einstein field equations), the energy equation. It describes the evolution of density ρ, ie. its rate of change $\dot{\rho}(t)$ as function of expansion factor $a(t)$. Derive this equation from the Friedmann equations.

f) There are 3 main groups of cosmic constituents, each with their own equation of state. Give each of these and specify their equation of state $p(\rho)$. In the case of dark energy, explain what the difference is between the cosmological constant pur sec and general dark energy.

g) For each of the cosmic constituents, infer the time evolution in terms of expansion factor $a(t)$.

h) Write out the Friedmann equation for the Hubble parameter $H(t)$, including only expansion factor $a(t)$ and Ω for the different cosmic constituents (ie. you will also have to write the curvature term according to what you found in d)).

i) Assume that only one of the four constituents is relevant at some point (I include curvature as one of these, ie. the case in which the total $\Omega = 0$ for the other constituents). Derive the four different solutions $a(t)$ for each of these cases: you may assume a flat $k = 0$ Universe for the radiation-dominated, matter-dominated and Lambda-dominated Universe. The one for a matter-dominated Universe is the Einstein-de Sitter Universe.

i) On the basis of the equation derived in h), you can obtain a qualitative expansion history of the Universe. Each of the 4 constituents (I am including curvature) may have dominated the expansion at some point in time. Make a sketch of the expansion history a(t). Indicate the points of possible dynamical transitions of the Universe (eg. radiation-matter transition), and provide expressions for the expansion factor $a(t)$ at which this transition occurs.

Question 4.

In question 3e we have looked at the Friedmann energy equation which describes the evolution of the energy density $\dot{\rho}$ of any constituent in the Universe.

a) Show that the derived equation in (3e) implies an adiabatically expanding medium. That is, demonstrate that the expansion of the Universe is adiabatic.

b) For an adiabatic expanding medium we know that $TV^{\gamma-1} = cst$, with T the temperature of the medium and V the volume. Derive the temperature change of a uniform radiation field in an expanding Universe as a function of expansion factor a_{exp}.

c) Given the fact that in an expanding Universe the frequency ν of radiation is redshifted, so that the frequency of a photon of current frequency ν_o has a frequency $\nu(z) = \nu_o(1+z)$ at redshift z, show that when the radiation field is blackbody at a particular cosmic epoch, it will remain blackbody ! (hint: combine temperature and frequency evolution of photons). What does this mean for the Cosmic Microwave Background.

d) It is known that to an impressive degree of accuracy the Cosmic Microwave Background is a black body radiation field, with a temperature of $T = 2.725K$

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp h\nu/kT - 1}$$

Infer the number density n_γ of photons as a function of the CMB temperature T. When compared to the number of baryons in the Universe, can you tell why the Universe is such an exceptional physical system ?

e) Also infer the energy density u_γ as a function of T. Why do n_γ and u_γ have different dependence on temperature T ?

f) The almost perfect blackbody character of the CMB has very strong implications for cosmology. What is this consequence ? When were most CMB photons created, ie. as a result of which cosmological process at around z 10^9 ? And which three radiation processes were responsible for transforming the spectral energy distribution of these primordial photons into a blackbody spectrum ? Why could simple Thomson scattering, so important at later times, not take care of this ?

g) At around a temperature of $T \approx 3000K$ the Universe, and in particular the blackbody cosmic radiation, undergoes a major transition. This, perhaps most important cosmological transition, includes three closely related processes. Describe each of these, and describe qualitatively what happened. You may use some drawings and sketches.

g) According to a simple equilibrium evaluation on the basis of the Saha equation the transition should have happened at a temperature of $T_\gamma = 3740K$. Why did it take place much later, at $T_\gamma \approx 3000K$?

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h) Even earlier in the Universe, at $z \approx 10^9$, the Universe is so hot that nuclear reactions can take place. Also here the Universe undergoes in a rapid sequence some three major transitions. How is it that at this stage the ratio between neutrons and protons gets fixed to around 1/5. Why does this later evolve into 1/6 ? Can you tell how this influences the final production of the light chemical elements ^4He ? What would you predict for the abundance of ^4He ?

Question 5.

Standard FRW cosmology has four important problems or "fine-tuning" issues. Inflationary cosmology, stating that the Universe underwent a rapid de Sitter expansion at a very early phase, tries to solve all four in one go.

a) List the four fundamental shortcomings of standard Big Bang cosmology. We will elaborate on the two most fundamental ones.

b) For the specific case of a curved matter-dominated Universe, work out the evolution of Ω as a function of expansion factor a (or redshift z) for various possible Universes. What will be the value of Ω at $a(t) = 0.001$ (recombination epoch) if the present $\Omega_0 = 0.3$. And at $a(t) \approx 0.0001$ (around matter-radiation equivalence). In general, what do you expect Ω to be for $a(t) \to 0$. Include a sketch of the Ω evolution. Explain this flatness problem !

c) What is the definition for the (particle) horizon in a FRW Universe. Give the expression for the horizon distance $d_{Hor}(t)$. How does the horizon distance evolve with time t in a radiation-dominated Universe (assume $\Omega_r = 1$), and in a matter-dominated Universe (assume $\Omega_m = 1$). How can it be that the horizon seems to grow faster than the velocity of light ? Subsequently, express these horizon distances in terms of the Hubble parameter H(t).

d) Work out the typical angular diameter θ of the horizon at recombination ($z_{dec} \approx 1000$).

- You may use the following expression for the angular diameter distance D_a in a matter-dominated Universe for an object at redshift z:

$$d_A = \frac{2c}{H_0\Omega_0^2}\frac{1}{(1+z)^2}\left\{\Omega_0 z + (\Omega_0 - 2)\sqrt{1+\Omega_0 z} - 1)\right\} \approx \frac{2c}{H_0\Omega_0}\frac{1}{z} \quad (\text{for } z \gg 1)$$

- if you are left with an equation including the Hubble parameter H(t), use the expression of the evolution of H(z) as function of redshift z (from FRW equation), approximating $(1+z) \approx z$ for $z \gg 1$ and using

$$\Omega_0 H_0^2 = \frac{8\pi G}{3}\rho_0$$

e) Given the impressive isotropy of the Cosmic Microwave Background, how does the derived value of $d_{Hor}(z_{dec})$ imply the "horizon" problem ?

f) Inflation is supposed to solve this situation. Explain how inflation does this. Specify the de Sitter expansion history during inflation, and how this may solve the three major Big Bang problems.

g) Which physical mechanism is responsible for inflation. Provide a description of the inflation potential, relate this to the various stages of inflation.

SUCCES !!!!
BEDANKT VOOR JULLIE AANDACHT EN INTERESSE DIT KWARTAIR !!!!

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